THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4240 - Stochastic Processes - 2023/24 Term 2

Reference Solutions to Homework 7
Due: Monday 22th April 2024

Exercises (Chapter 3, Page 107): 11, 12, 13, 16, 17, 18, 19, 21,

and

Q1. Suppose that the time to repair a machine is an exponentially distributed random variable with mean 2h.
(a) What is the probability that the repair takes more than 2h?
(b) What is the probability that the repair takes more than 5h given that it has taken more than 3h?

Q2. Alice and Betty enter a beauty parlor simultaneously, Alice to get a manicure and Betty to get a haircut. Suppose that the time for a manicure (haircut) is exponentially distributed with mean 20 (30) min.
(a) What is the probability that Alice gets done first?
(b) What is the expected amount of time until Alice and Betty are both done?

Q3. Ron, Sue, and Ted arrive at the beginning of a professor's office hours. The amount of time they will stay is exponentially distributed with means of 1, 1/2, and 1/3 hour.
(a) What is the expected time until only one student remains?
(b) For each student find the probability they are the last student left.
(c) What is the expected time until all three students are gone?

Q4. A telephone booth has 1 telephone and 2 waiting spaces. Suppose people come in as a Poisson process with rate 2 per minute. Each one use the phone for 1 minute in average, and the usage time is an exponential random variable. Let $X(t)$ denote the number of people in the booth at time t.
(a) Find the rate matrix for the process.
(b) In the long term, what is the probability that there are 2 persons in the booth?

Q5. Suppose that the arrival rate at a checkout counter is 2 customers per minute. A single clerk is working at the counter and the service time is an exponential random variable with mean time 1/2 minute. However, if there are 3 customers or more, then someone will come to help and the service time reduces to a mean of 1/3 minute.

(a) Set up the queuing model in an infinite matrix.
(b) What is the stationary distribution of the queue?
(c) In the long term, what is probability that there are 4 customers waiting (including the one being served)?

Q6. Suppose there are three computers in an office that are subject to failure and repair. The failure of each computer is an exponential distribution with average once in 50 days; there is only on repairman and the repair time for each computer is an exponential distribution with mean 2 days. In the long term, what is the probability that all three computers are functioning?

11. Solution. Note that in page 101, $X(t) = X_1(t) + X_2(t)$, where $X_1(t)$ is the number of customers arriving in $(0, t]$ that are still in the process of being served at time t, and $X_2(t)$ is the number of the initial x customers still in the process of being served at time t. Moreover, $X_1(t)$ has a Poisson distribution with parameter $\frac{\lambda}{\mu}(1-e^{-\mu t})$; $X_2(t)$ is independent of $X_1(t)$ and has a binomial distribution with parameters x and $e^{-\mu t}$. Hence

$$E(X(t)) = E(X_1(t)) + E(X_2(t)) = \frac{\lambda}{\mu}(1 - e^{-\mu t}) + xe^{-\mu t}.$$

and

$$\begin{aligned}\operatorname{Var}(X(t)) = \operatorname{Var}(X_1(t)) + \operatorname{Var}(X_2(t)) &= \frac{\lambda}{\mu}(1 - e^{-\mu t}) + xe^{-\mu t}(1 - e^{-\mu t}) \\ &= \left(\frac{\lambda}{\mu} + xe^{-\mu t}\right)(1 - e^{-\mu t}).\end{aligned}$$

12. Solution. (a) The forward equation is

$$\begin{cases} P'_{x0}(t) = \mu P_{x1}(t), \\ P'_{xy}(t) = \lambda(y-1)P_{x,y-1}(t) - (\lambda+\mu)yP_{xy}(t) + \mu(y+1)P_{x,y+1}(t), \quad y \geq 1. \end{cases}$$

(b) By the forward equation,

$$\begin{aligned} m'_x(t) &= \sum_{y=0}^{\infty} yP'_{xy}(t) = \sum_{y=1}^{\infty} yP'_{xy}(t) \\ &= \sum_{y=1}^{\infty} \big(\lambda y(y-1)P_{x,y-1}(t) - (\lambda+\mu)y^2P_{xy}(t) + \mu y(y+1)P_{x,y+1}(t)\big) \\ &= \sum_{y=0}^{\infty} \lambda(y+1)yP_{xy}(t) - \sum_{y=1}^{\infty}(\lambda+\mu)y^2P_{xy}(t) + \sum_{y=2}^{\infty} \mu(y-1)yP_{xy}(t) \\ &= \sum_{y=0}^{\infty} \big(\lambda(y+1)y - (\lambda+\mu)y^2 + \mu(y-1)y)\big)P_{xy}(t) \\ &= \sum_{y=0}^{\infty}(\lambda-\mu)yP_{xy}(t) = (\lambda-\mu)m_x(t). \end{aligned}$$

(c) Solving the ODE in (b) under the initial condition

$$m_x(0) = \sum_{y=0}^{\infty} yP_{xy}(0) = \sum_{y=0}^{\infty} \delta_{xy} \cdot y = x,$$

we get

$$m_x(t) = m_x(0)e^{(\lambda-\mu)t} = xe^{(\lambda-\mu)t}.$$

13. Solution. **(a)** By the forward equation,

$$\begin{aligned}
s_x'(t) &= \sum_{y=0}^{\infty} y^2 P_{xy}'(t) = \sum_{y=1}^{\infty} y^2 P_{xy}'(t) \\
&= \sum_{y=1}^{\infty} \big(\lambda y^2(y-1)P_{x,y-1}(t) - (\lambda+\mu)y^3 P_{xy}(t) + \mu y^2(y+1)P_{x,y+1}(t)\big) \\
&= \sum_{y=0}^{\infty} \lambda(y+1)^2 y P_{xy}(t) - \sum_{y=1}^{\infty}(\lambda+\mu)y^3 P_{xy}(t) + \sum_{y=2}^{\infty} \mu(y-1)^2 y P_{xy}(t) \\
&= \sum_{y=0}^{\infty} \big(\lambda(y+1)^2 y - (\lambda+\mu)y^3 + \mu(y-1)^2 y)\big) P_{xy}(t) \\
&= \sum_{y=0}^{\infty} \big(2(\lambda-\mu)y^2 + (\lambda+\mu)\big) P_{xy}(t) \\
&= 2(\lambda-\mu)s_x(t) + (\lambda+\mu)m_x(t) \\
&= 2(\lambda-\mu)s_x(t) + (\lambda+\mu)xe^{(\lambda-\mu)t}.
\end{aligned}$$

(b) Solving the ODE in (a) under the initial condition

$$s_x(0) = \sum_{y=0}^{\infty} y^2 P_{xy}(0) = \sum_{y=0}^{\infty} \delta_{xy} \cdot y^2 = x^2,$$

we get

$$s_x(t) = \begin{cases} \left(x^2 + \dfrac{\lambda+\mu}{\lambda-\mu}x\right) e^{2(\lambda-\mu)t} - \dfrac{\lambda+\mu}{\lambda-\mu} x e^{(\lambda-\mu)t}, & \lambda \neq \mu, \\ x^2 + 2\lambda x t, & \lambda = \mu. \end{cases}$$

(c) Under the condition $X(0) = x$,

$$\mathrm{Var}X(t) = s_x(t) - (m_x(t))^2 = \begin{cases} \dfrac{\lambda+\mu}{\lambda-\mu} x(e^{2(\lambda-\mu)t} - e^{(\lambda-\mu)t}), & \lambda \neq \mu, \\ 2\lambda x t, & \lambda = \mu. \end{cases}$$

16. Solution. We use the criterion (51) and (56) in textbook.

(a) Note that

$$\sum_{x=1}^{\infty} \frac{\mu_1 \cdots \mu_x}{\lambda_1 \cdots \lambda_x} = \sum_{x=1}^{\infty} \frac{x!}{(x+1)!} = \sum_{x=1}^{\infty} \frac{1}{x+1} = \infty,$$

and

$$\sum_{x=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{x-1}}{\mu_1 \cdots \mu_x} = \sum_{x=1}^{\infty} \frac{x!}{x!} = \sum_{x=1}^{\infty} 1 = \infty.$$

Hence the process is null recurrent.

(b) Note that

$$\sum_{x=1}^{\infty} \frac{\mu_1 \cdots \mu_x}{\lambda_1 \cdots \lambda_x} = \sum_{x=1}^{\infty} \frac{x!}{(x+2)!} = \sum_{x=1}^{\infty} \frac{1}{(x+1)(x+2)} = \sum_{x=1}^{\infty} \left(\frac{1}{x+1} - \frac{1}{x+2}\right) = \frac{1}{2} < \infty.$$

Hence the process is transient.

17. Proof. Consider the embedded Markov chain (in page 102 of textbook) with transition function

$$P(x,y) = Q_{xy} = \begin{cases} 1, & x=0,\ y=1; \\ \dfrac{\lambda_x}{\lambda_x+\mu_x} = p_x, & y=x+1,\ x\geq 1; \\ \dfrac{\mu_x}{\lambda_x+\mu_x} = q_x, & y=x-1,\ x\geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

We see that the embedded chain is a birth and death chain on the nonnegative integers. Moreover, $\gamma_0 = 1$, and for $x \geq 1$,

$$\gamma_x = \frac{q_1 \cdots q_x}{p_1 \cdots p_x} = \frac{\mu_1 \cdots \mu_x}{\lambda_1 \cdots \lambda_x}.$$

Using Q26 of Chapter 1, we get (a) and (b) immediately.

18. Proof. (a) Note that $\gamma_y = (\frac{\mu}{\lambda})^y$. As now $\mu \geq \lambda$, so $\sum_y \gamma_y = \infty$. Hence $\rho_{x0} = 1$ by Q17(a).

(b) If $\mu < \lambda$, by Q17(b),

$$\rho_{x=0} = \frac{\sum_{y=x}^{\infty}(\mu/\lambda)^y}{\sum_{y=0}^{\infty}(\mu/\lambda)^y} = \left(\frac{\mu}{\lambda}\right)^x, \quad x \geq 1.$$

19. Proof. Note that $\gamma_y = (\frac{1-p}{p})^y$. If $p \leq \frac{1}{2}$, then $\sum_y \gamma_y = \infty$. Hence $\rho_{x0} = 1$ by Q17(a).

If $p > \frac{1}{2}$, then by Q17(b),

$$\rho_{x0} = \frac{\sum_{y=x}^{\infty}(\frac{1-p}{p})^y}{\sum_{y=0}^{\infty}(\frac{1-p}{p})^y} = \left(\frac{1-p}{p}\right)^x, \quad x \geq 1.$$

21. Solution. Using the result in page 105 of textbook, set $\pi_0 = 1$, and

$$\pi_x = \frac{\lambda_0 \cdots \lambda_{x-1}}{\mu_1 \cdots \mu_x} = \frac{1}{x!}\left(\frac{\lambda}{\mu}\right)^x, \quad 1 \leq x \leq d.$$

Then the stationary distribution is given by

$$\pi(x) = \frac{\pi_x}{\sum_{y=0}^{d} \pi_y} = \frac{\frac{(\lambda/\mu)^x}{x!}}{\sum_{y=0}^{d} \frac{(\lambda/\mu)^y}{y!}}, \quad 0 \leq x \leq d.$$

SQ1. Let X be the time to repair a machine. Note that $X \sim Exp(\frac{1}{2})$.
(a)

$$\begin{aligned} P(X > 2) &= 1 - P(X \leq 2) \\ &= 1 - (1 - e^{-\frac{1}{2}\cdot 2}) = e^{-1}. \end{aligned}$$

Instead of memorizing the formula of $P(X \leq t)$ for exponential random variable X, one can memorize its probability density function and do calculation. Recall

$$f_X(t) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}t} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Since exponential random variable is memoryless, hence by part (a), we have

$$P(X > 5|X > 3) = P(X > 2) = e^{-1}.$$

SQ2. Let X_1 be the time for Alice to get done, and X_2 be the time for Betty. Note that $X_1 \sim Exp(\frac{1}{20})$ and $X_2 \sim Exp(\frac{1}{30})$.

(a) Then, by (19) on p.90 of our textbook, we have

$$P(X_1 < X_2) = \frac{\frac{1}{20}}{\frac{1}{20} + \frac{1}{30}} = \frac{3}{5}.$$

This can also be verified by the following calculation (recall that X_1, X_2 are independent):

$$\begin{aligned} P(X_1 < X_2) &= \int_0^\infty P(X_1 < t | X_2 = t) f_{X_2}(t) dt \\ &= \int_0^\infty (1 - e^{-\lambda_1 t}) \lambda_2 e^{-\lambda_2 t} dt \\ &= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2}, \end{aligned}$$

where λ_1, λ_2 are the rates of X_1, X_2. ($\lambda_1 = 1/20, \lambda_2 = 1/30$.)

(b) Let $X = \max\{X_1, X_2\}$. Then, by independence of the two r.v.'s

$$\begin{aligned} P(X < t) &= P(X_1 < t, X_2 < t) \\ &= P(X_1 < t) P(X_2 < t) \\ &= (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t}) \\ &= 1 - e^{-\lambda_1 t} - e^{-\lambda_2 t} + e^{-(\lambda_1 + \lambda_2) t}. \end{aligned}$$

Hence, $f_X(t) = \lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2) t}$ for $t \geq 0$. The expected amount of time until Alice and Betty are both done is

$$E(X) = \int_0^\infty t f_X(t) dt = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2} = 38.$$

Another approach to do this question is by considering

$$X_1 + X_2 = \max\{X_1, X_2\} + \min\{X_1, X_2\}.$$

Taking $E(\cdot)$ on both sides, and recall that if X_1 and X_2 are independent exponential random variables with rates λ_1, λ_2, then $\min\{X_1, X_2\}$ is an exponential random variable with rate $\lambda_1 + \lambda_2$, we obtain

$$20 + 30 = E(X) + \frac{1}{\frac{1}{20} + \frac{1}{30}}$$

SQ3. Let X_1, X_2, X_3 be the amount of time that Ron, Sue, Ted will stay in the office. Note that $X_1 \sim Exp(1), X_2 \sim Exp(2), X_3 \sim Exp(3)$, and they are independent.

(a). We may condition on whos is the first one to leave. For example, if Ron is the first one to leave, then the expected time until only one student remains is $\min\{X_2, X_3\}$. Let F_i be the event that $\{X_i = \min\{X_1, X_2, X_3\}\}$. Then, the expected time is given by

$$E[\min\{X_2, X_3\}|F_1]P(F_1) + E[\min\{X_3, X_1\}|F_2]P(F_2) + E[\min\{X_1, X_2\}|F_3]P(F_3).$$

As an example, we may calculate $E[\min\{X_2, X_3\}|F_1]P(F_1)$. Note that for $t > 0$,

$$\begin{aligned}
E[\min\{X_2, X_3\}|F_1]P(F_1) &= \frac{1}{P(F_1)} P(X_1 < \min\{X_2, X_3\} < t) \\
&= \frac{1}{P(F_1)} \int_0^t P(X_1 < s) f_{\min\{X_2, X_3\}}(s) ds \\
&= \frac{1}{P(F_1)} \int_0^t (1 - e^{-\lambda_1 s})(\lambda_2 + \lambda_3) e^{-(\lambda_2+\lambda_3)s} ds \\
&= \frac{\lambda_2 + \lambda_3}{P(F_1)} \int_0^t e^{-(\lambda_2+\lambda_3)s} - e^{-(\lambda_1+\lambda_2+\lambda_3)s} ds \\
&= 1 - \frac{\lambda_1 + \lambda_2 + \lambda_3}{\lambda_1} e^{-(\lambda_2+\lambda_3)t} + \frac{\lambda_2 + \lambda_3}{\lambda_1} e^{-(\lambda_1+\lambda_2+\lambda_3)t}.
\end{aligned}$$

The last step uses the fact that $P(F_1) = \lambda_1/(\lambda_1 + \lambda_2 + \lambda_3)$. One may calculate the average time by the formula

$$E[\min\{X_2, X_3\}|F_1] = \int_0^\infty P(\min\{X_2, X_3\} > t|F_1) dt.$$

This is legitimate only when the random variable X satisfies $X \geq 0$. Or one can first find the probability density function $f_X(t)$ and integrate $\int_0^\infty t f(t) dt$. Hence,

$$\begin{aligned}
E[\min\{X_2, X_3\}|F_1] &= \frac{\lambda_1 + \lambda_2 + \lambda_3}{\lambda_1(\lambda_2 + \lambda_3)} - \frac{\lambda_2 + \lambda_3}{\lambda_1(\lambda_1 + \lambda_2 + \lambda_3)} \\
&= \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_2 + \lambda_3} = \frac{11}{30}.
\end{aligned}$$

Using the formula above, one can show that $E[\min\{X_3, X_1\}|F_2] = 5/12$ and $E[\min\{X_1, X_2\}|F_3] = 5/12$. Therefore, the required expected amount of time is

$$\frac{11}{30} \cdot \frac{1}{6} + \frac{5}{12} \cdot \frac{2}{6} + \frac{1}{2} \cdot \frac{3}{6} = \frac{9}{20}.$$

(b). Since X_2 and X_3 are independent, the probability that Ron is the last student to left is

$$\begin{aligned}
P(X_1 > \max\{X_2, X_3\}) &= \int_0^\infty P(\max\{X_2, X_3\} < t) f_{X_1}(t) dt \\
&= \int_0^\infty P(X_2 < t) P(X_3 < t) f_{X_1}(t) dt \\
&= \int_0^\infty (1 - e^{-\lambda_2 t})(1 - e^{-\lambda_3 t}) \lambda_1 e^{-\lambda_1 t} dt \\
&= 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2} - \frac{\lambda_1}{\lambda_1 + \lambda_3} + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{7}{12}.
\end{aligned}$$

Using the formula above, we have $P(X_2 > \max\{X_3, X_1\}) = 4/15$ and $P(X_3 > \max\{X_1, X_2\}) = 3/20$.

(c). The required expected value is $E[\max\{X_1, X_2, X_3\}]$. Note that for $t > 0$, since X_1, X_2, X_3 are independent, we have

$$\begin{aligned} P(\max\{X_1, X_2, X_3\} < t) &= P(X_1 < t)P(X_2 < t)P(X_3 < t) \\ &= (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})(1 - e^{-\lambda_3 t}) \\ &= 1 - e^{-\lambda_1 t} - e^{-\lambda_2 t} - e^{-\lambda_3 t} + e^{-(\lambda_1+\lambda_2)t} + e^{-(\lambda_2+\lambda_3)t} + e^{-(\lambda_3+\lambda_1)t} - e^{-(\lambda_1+\lambda_2+\lambda_3)t} \end{aligned}$$

$$\begin{aligned} E[\max\{X_1, X_2, X_3\}] &= \int_0^\infty P(\max\{X_1, X_2, X_3\} > t)dt \\ &= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} - \frac{1}{\lambda_1+\lambda_2} - \frac{1}{\lambda_2+\lambda_3} - \frac{1}{\lambda_3+\lambda_1} + \frac{1}{\lambda_1+\lambda_2+\lambda_3} \\ &= \frac{73}{60}. \end{aligned}$$

SQ4 (a). The rate matrix is given by

$$D = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} -2 & 2 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array}.$$

(b). Note that for birth and death process, if we put

$$\pi_x = \frac{\lambda_0 \dots \lambda_{x-1}}{\mu_1 \dots \mu_x} \text{ for } x = 1, 2 \text{ and } 3,$$

then the stationary distribution π is given by

$$\pi(x) = \begin{cases} (1 + \sum_{y=1}^{3} \pi_y)^{-1} & \text{if } x = 0, \\ \pi_x(1 + \sum_{y=1}^{3} \pi_y)^{-1} & \text{if } x = 1, 2 \text{ or } 3. \end{cases}$$

Since $\lambda_0 = \lambda_1 = \lambda_2$ and $\mu_1 = \mu_2 = \mu_3 = 1$, it follows that

$$\pi = (1/15, 2/15, 4/15, 8/15).$$

The required probability is $\lim_{t\to\infty} P(X(t) = 2) = \pi(2) = 4/15$.

SQ5. Let $X(t)$ be the number of customers that we are waiting or being served at time t.

(a). The infinite matrix is given by

$$D = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & \dots \end{matrix} \\ \begin{bmatrix} -2 & 2 & & & & & \\ 2 & -4 & 2 & & & & \\ & 2 & -4 & 2 & & & \\ & & 3 & -5 & 2 & & \\ & & & 3 & -5 & 2 & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix} \end{array}.$$

(b). Note that $\lambda_x = 2$ for all $x \geq 0$ and

$$\mu_x = \begin{cases} 2 & \text{if } 1 \leq x \leq 2, \\ 3 & \text{if } x \geq 3. \end{cases}$$

So,

$$\pi_x = \begin{cases} 1 & \text{if } 1 \leq x \leq 2 \\ (\frac{2}{3})^{x-2} & \text{if } x \geq 3. \end{cases}$$

Let π be the stationary distribution of this birth and death process. Since $\sum_{y=1}^{\infty} \pi_y = 4$, we have

$$\pi(x) = \begin{cases} \frac{1}{5} & \text{if } 0 \leq x \leq 2, \\ \frac{1}{5}(\frac{2}{3})^{x-2} & \text{if } x \geq 3. \end{cases}$$

(c).

$$\lim_{t \to \infty} P(X(t) = 4) = \pi(4) = \frac{4}{45}.$$

SQ6. Let $X(t)$ be the number of working computers at time t. The rate matrix is given by

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ D = & \begin{bmatrix} -1/2 & 1/2 & 0 & 0 \\ 1/50 & -13/25 & 1/2 & 0 \\ 0 & 2/50 & -27/50 & 1/2 \\ 0 & 0 & 3/50 & -3/50 \end{bmatrix}. \end{array}$$

This is a birth and death process with finitely many states. We can find its stationary distribution by formula. Note that

$$\begin{aligned} \pi_1 &= \frac{\frac{1}{2}}{\frac{1}{50}} = 25 \\ \pi_2 &= \frac{(\frac{1}{2})^2}{\frac{1}{50}\frac{2}{50}} = \frac{625}{2} \\ \pi_3 &= \frac{(\frac{1}{2})^3}{\frac{1}{50}\frac{2}{50}\frac{3}{50}} = \frac{15625}{6} \end{aligned}$$

$$1 + \sum_{y=1}^{3} \pi_y = \tfrac{8828}{3}$$

Therefore, the stationary distribution π is $(3/8828, 75/8828, 1875/17656, 15625/17656)$. The required probability is

$$\lim_{t\to\infty} P(X(t) = 3) = \pi(3) = \frac{15625}{17656}.$$